Functions of the first Baire class

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We aim at starting an analysis of definable functions similar to the Wadge theory for definable sets, focusing more specifically on Baire class 1 functions between 0-dimensional Polish spaces. To parallel Wadge's analysis, we break this study in two parts. The first concerns subclasses of the first Baire class characterisable by infinite games, while the second looks at the quasi-order of continuous reducibility on continuous functions.

Here X, Y, X' and Y' are variables for Polish 0-dimensional (P0D for short) spaces, considered as closed subspaces of the Baire space of infinite sequences of natural numbers.

Playing in the first Baire class. [1] We consider various infinite games with a function $f : X \to Y$ as parameter. The simplest one sees Player I choosing a natural number x_i and Player II a natural y_i at the i-th round, thus building two infinite sequences x and y. Player II then wins the game if and only if $f(x) = y$. In this game, Player II has a winning strategy if and only if f is Lipschitz; we say that the game *characterises* Lipschitz functions.

New rules giving more and more power to Player II will then characterise larger and larger classes of functions. For instance, allowing Player II to skip her turn as often as she wants characterises continuous functions. These first two games were defined by Wadge in his PhD thesis. We consider three others.

In the *backtrack* game, defined by van Wesep in his PhD thesis, Player II can erase what she did at the previous round, but only finitely many times during a run. It characterises functions that are σ-continuous with closed witnesses.

Duparc in his PhD thesis defined the *eraser* game by allowing Player II to erase infinitely often. This one characterises Baire class 1 functions.

The third one is a refinement of the first one, defined by Motto Ros in his PhD thesis. It is called the α*-bounded backtrack* game, because the erasing ability of II is bounded by some countable ordinal α.

We prove that all three games are determined. Observe that instead of considering only Borel functions and use Martin's result on Borel determinacy, we show a stronger determinacy result.

Theorem 1. *For all functions $f : X \to Y$, the eraser game, the backtrack game and the α-bounded backtrack game with parameter f are determined.*

We in fact define the finer notion of an *aggressive* strategy for Player I to get a stronger theorem; namely if II has no winning strategy in the backtrack game with parameter $f : X \to Y$, then I has an aggressive winning strategy.

As a corollary, we give a new proof, purely game-theoretic, of the Baire Lemma on pointwise convergence.

[1] This was published in *Mathematical Logic Quarterly*, vol. 60, n. 1-2, pp. 118-132, 2014

A quasi-order for continuous functions. [2] Given two functions $f : X \to Y$ and $g : X' \to Y'$, we write $f \leq g$ and say that g *continuously reduces* f if there are two continuous functions $\sigma : X \to X'$ and $\tau : \mathrm{Im}(g \circ \sigma) \to Y$ satisfying $f = \tau \circ g \circ \sigma$. We write $f \equiv g$ when both $f \leq g$ and $g \leq f$ hold.

Computable versions of this quasi-order, introduced by Weihrauch, have become central in computable analysis.

We first prove that any two continuous functions $f : X \to Y$ and $g : X' \to Y'$ with uncountable ranges are continuously bi-reducible. We then study more specifically functions with countable image; we let both C_∞ denote the class of all continuous functions between P0D spaces and C denote the subclass of C_∞ of functions with countable range.

We define the *Cantor-Bendixson rank* $\mathrm{CB}(f)$ *of a function* f in C. This rank generalises the usual Cantor-Bendixson rank on closed sets, in the sense that $\mathrm{CB}(\mathrm{Id}_F) = \mathrm{CB}(F)$ for all closed F. This rank stratifies C in classes C_α of all functions in C of Cantor-Bendixson rank α, for α countable.

We isolate two infinitary operations on P0D spaces, called the *gluing* and the *pointed gluing*. We give universal properties for these, and prove that they generate, from the empty set, all countable P0D spaces up to homeomorphism.

We then define the gluing and pointed gluing on sequences of functions so that the operations on sets and those on functions commute with the identity functor.

Letting $\mathsf{C}^\star$ be the subclass of C of functions with compact domain, we prove

Theorem 2. *The relation $\leq$ is a well-order on $\mathsf{C}^\star / \equiv$ of order type ω_1.*

Using $\mathsf{C}^\star$ as a leverage point in C, we describe the general structure of $(\mathsf{C}, \leq)$.

Theorem 3. *Given f, g in C, $\lambda < \omega_1$ a limit ordinal and n a natural, we have*

1. *if $\mathrm{CB}(f) = \mathrm{CB}(g) = \lambda$ then $f \equiv g$,*
2. *if $\mathrm{CB}(f) = \lambda + n$ and $\mathrm{CB}(g) = \lambda + 2n + 1$, then $f \leq g$.*

In particular, if $(\mathsf{C}_\alpha, \leq)$ is a well-quasi-order (wqo) for all $\alpha \in \omega_1$ then $(\mathsf{C}_\infty, \leq)$ is a wqo. The question of wether C_∞ is a wqo or not remains open.

Applying Theorem 3 to the subclass of identity functions, we obtain an alternative proof that topological embeddability on P0D spaces is a wqo. This result is also obtained as a corollary of Laver's celebrated result on labelled trees.

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[2]This was published in the *Journal of Symbolic Logic*, vol. 78, n. 2, pp. 633-648, 2013